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An algorithm for computing moments-based flood quantile estimates when historical flood information is available

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Abstract. This paper presents the expected moments algorithm (EMA), a simple and efficient method for incorporating historical and paleoflood information into flood frequency studies. EMA can utilize three types of at-site flood information: systematic stream gage record; information about the magnitude of historical floods; and knowledge of the number of years in the historical period when no large flood occurred. EMA employs an iterative procedure to compute method-of-moments parameter estimates. Initial parameter estimates are calculated from systematic stream gage data. These moments are then updated by including the measured historical peaks and the expected moments, given the previously estimated parameters, of the below-threshold floods from the historical period. The updated moments result in new parameter estimates, and the last two steps are repeated until the algorithm converges. Monte Carlo simulations compare EMA, *Bulletin 17B*'s [*United States Water Resources Council*, 1982] historically weighted moments adjustment, and maximum likelihood estimators when fitting the three parameters of the log-Pearson type III distribution. These simulations demonstrate that EMA is more efficient than the *Bulletin 17B* method, and that it is nearly as efficient as maximum likelihood estimation (MLE). The experiments also suggest that EMA has two advantages over MLE when dealing with the log-Pearson type III distribution: It appears that EMA estimates always exist and that they are unique, although neither result has been proven. EMA can be used with binomial or interval-censored data and with any distributional family amenable to method-of-moments estimation.

1. Introduction

This paper presents the expected moments algorithm (EMA) as an alternative to maximum likelihood estimation (MLE) or the *Bulletin 17B* (B17) [*United States Water Resources Council*, 1982] method for incorporating historical information in flood frequency studies. Use of historical information to improve flood quantile estimates has been investigated previously [*Leese*, 1973; *Tasker and Thomas*, 1978; *Condie and Lee*, 1982; *Condie and Pilon*, 1983; *Condie*, 1986; *Stedinger and Cohn*, 1986; *Hosking and Wallis*, 1986a, b; *Cohn and Stedinger*, 1987; *Lane*, 1987; *Stedinger and Cohn*, 1987; *Jin and Stedinger*, 1989; *Wang*, 1990a, b; *Guo and Cunnane*, 1991; *Kuczera*, 1992; *Pilon and Adamowski*, 1993; *Frances and Salas*, 1994].

As with *Stedinger and Cohn* [1986], we assume there is an historical period of N_H years. The historical period is a span of time for which no systematic gage measurements are available yet for which some inference may be made about the flood history from other sources. For example, historical information may be available from newspaper accounts of extraordinary floods or from flood lines on buildings. The beginning of the "historical period" might correspond to the first year that settlers living next to the river would have noted an extraordinarily large flood. In most cases the historical period would end when a stream gage was installed. During the N_H-year historical period, the magnitudes of $N_H^>$ (possibly 0) floods were recorded because they were unusually large, greater than a known threshold, T. T would correspond to the discharge above which some sort of permanent flood record would be created, perhaps corresponding to inundation of a town's main street. Analogous to the historical peaks, there were also $N_H^<$ unmeasured annual peak floods with magnitudes less than T. Because the magnitudes of these small annual peaks were not recorded (except insofar as they did not exceed T), they are treated as type I censored observations. Finally, it is assumed that the historical period is followed by N_S years of systematic gage record, during which $N_S^>$ floods were greater than T and $N_S^<$ floods were less than T. The magnitudes of all N_S systematic-record floods are known. The total record length, N, is equal to $N_S + N_H$. Figure 1 illustrates these terms. (Note that the definitions here differ from those of B17).

We denote annual peak floods as $\{Q_1, \cdots, Q_N\}$ and their logarithms as $\{X_1, \cdots, X_N\}$. We assume the logarithms are independent and identically distributed and that they obey a Pearson type III (P-III) distribution with parameters (α, β, τ). The properties of this distribution are well known [*Matalas and Wallis*, 1973; *Bobee*, 1975; *Bobee and Robitaille*, 1977; *Condie*,

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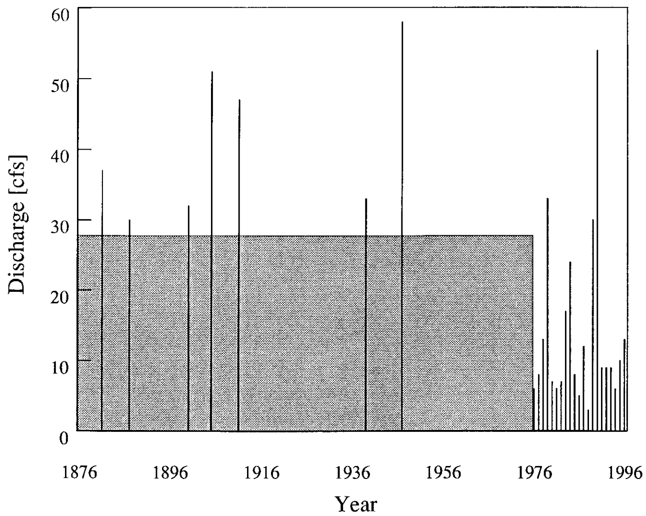

Figure 1. A flood record with both historical and systematic data. $T = 28$ (cubic feet per second), $N = 120$, $N_H = 100$, $N_H^> = 7$, $N_H^< = 93$, $N_S = 20$, $N_S^> = 3$, and $N_S^< = 17$. The gray strip indicates censoring during the historical period.

1977; *Lall and Beard*, 1982; *Bowman and Shenton*, 1988; *Kite*, 1988]. The P-III probability density function is given by

$$f(x|\alpha,\ \beta,\ \tau) = \frac{\left(\dfrac{x-\tau}{\beta}\right)^{a-1} \exp\left(-\dfrac{x-\tau}{\beta}\right)}{|\beta|\Gamma(\alpha)} \qquad \left(\frac{x-\tau}{\beta}\right) \geq 0$$

$$f(x|\alpha,\ \beta,\ \tau) = 0 \qquad \text{otherwise} \tag{1}$$

where

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp\,(-t)\; dt \tag{2}$$

Using superscripts and subscripts as above, the set $\{X\}$ can be expressed as the union of four sets:

$$\{X\} = \{X_S^>\} \cup \{X_H^>\} \cup \{X_S^<\} \cup \{X_H^<\} \tag{3}$$

where

$\{X_S^>\}$ logarithms of floods greater than T which occurred during the systematic record and whose magnitudes were measured by stream gage;

$\{X_H^>\}$ logarithms of "historical floods" greater than T that occurred during the historical period;

$\{X_S^<\}$ logarithms of systematic-record floods less than T, measured by stream gage;

$\{X_H^<\}$ logarithms of historical-period floods smaller than T that were not measured at all, except insofar as we know their magnitudes did not exceed the threshold T.

2. Estimating Parameters of the Log-Pearson Type III Distribution

B17 [also see *Tasker and Thomas*, 1978] recommends using the method of moments to fit the parameters of the P-III distribution to the logarithms of annual peak discharges. (B17 also recommends using a regionalized skew coefficient; to simplify the discussion, regionalized skew is not considered here). Sample moments of $\{X\}$ must be computed. But how should one incorporate into the analysis the $N_H^<$ below-threshold floods whose magnitudes were not recorded? Appendix 6 of B17 recommends using moments from the $N_S^<$ below-threshold observations from the systematic record to represent the moments of the $\{X_H^<\}$ floods. Their composite value is estimated by use of a weighting factor

$$W = \frac{N^<}{N_S^<} \tag{4}$$

to compute adjusted moments:

$$\hat{\mu} = \frac{W \Sigma X_S^< + \Sigma X^>}{N} \tag{5}$$

$$\hat{\sigma}^2 = \frac{W \Sigma\,(X_S^< - \hat{\mu})^2 + \Sigma\,(X^> - \hat{\mu})^2}{N-1} \tag{6}$$

$$\hat{\gamma} = \frac{N(W \Sigma\,(X_S^< - \hat{\mu})^3 + \Sigma\,(X^> - \hat{\mu})^3)}{(N-1)(N-2)\hat{\sigma}^3} \tag{7}$$

where the circumflexes indicate that these parameters are estimators. The P-III distribution's parameters are then estimated by equating the first three sample moments to the distribution's moments and solving for $(\alpha,\ \beta,\ \tau)$

$$\hat{\alpha} = \frac{4}{\hat{\gamma}^2} \tag{8}$$

$$\hat{\beta} = \text{sign}\,(\hat{\gamma})\left(\frac{\hat{\sigma}^2}{\hat{\alpha}}\right)^{1/2} \tag{9}$$

$$\hat{\tau} = \hat{\mu} - \hat{\alpha}\hat{\beta} \tag{10}$$

The pth quantile of the fitted distribution is estimated by

$$\hat{X}_p = \hat{\tau} + \hat{\beta}P^{-1}(\hat{\alpha},\ p) \tag{11}$$

where $P^{-1}(\hat{\alpha},\ p)$ is the inverse of the incomplete Gamma function [*Abramowitz and Stegun*, 1964]. In effect, each of the below-threshold observations from the systematic record is given increased weight to represent the unobserved values, $\{X_H^<\}$. A similar weighting approach is employed by *Wang* [1990a, b] to compute L-moments-based flood estimates.

Apparently, the B17 historic adjustment was developed to address two specific circumstances that can arise in flood frequency studies: (1) the presence of a huge flood in a relatively short systematic gage record (for example, a community experiencing a flood may know that it is the largest to have occurred in a hundred years, even though the gage record extends back only 20 years old) and (2) the knowledge that a very large flood had occurred, usually prior to the beginning of systematic gaging, that was not part of the systematic gage record.

It can be inferred from the B17 method and from the discussion in B17 that the B17 adjustment was designed primarily to make flood-quantile estimates consistent with "community experience" in such cases (W. H. Kirby, oral communication, 1995).

Recently, *Stedinger and Cohn* [1986] showed that much of the information contained in an historical flood record is connected with knowing the number of exceedances of the threshold rather than in knowing the magnitudes of the "historic" floods. Even the knowledge that no "historic" floods occurred during the historical period provides information that can improve the accuracy of flood quantile estimates. The B17 adjustment was not designed to make use of threshold-exceedance information, and thus alternative methods are needed to exploit this kind of data.

One alternative method is likelihood-based fitting, which has been found to be statistically efficient, though often computationally difficult, for many distributions [*Bobee*, 1975; *Bobee and Robitaille*, 1977; *Condie*, 1977; *Bowman and Shenton*, 1988; *Stedinger and Cohn*, 1986]. Fitting the P-III distribution by maximum likelihood requires a numerical search for a local maximum of the likelihood function (the global maximum corresponds to nonsense estimates; for $\alpha < 1$ the likelihood function is unbounded in the limit as τ approaches the smallest (largest when $\beta < 0$) observation). Condie and Pilon [*Condie and Pilon*, 1983; *Condie*, 1986] describe an interval-bisection algorithm following algebraic simplification of the likelihood equations and report encountering no failures to obtain MLE estimates in a Monte Carlo study. However, even without the complication of censored data, many researchers have had trouble fitting the P-III distribution by maximum likelihood. *Bowman and Shenton* [1988, p. 155] note that the properties of "maximum likelihood estimators perhaps have little advantage over moment estimators from the viewpoint of the existence of moments." Researchers have had difficulty finding any plausible local maximum [*Rao*, 1986] or report failures in their numerical optimization methods [*Matalas and Wallis*, 1973]. *Hirose* [1995] presents an algorithm that reportedly finds all local MLEs wherever they exist. However, he reports that frequently no local MLE exists for small samples. Hirose also shows that more than one local maximum can exist for a given sample. In this paper we present a simple and efficient moments-based alternative that avoids these complications.

3. The Expected Moments Algorithm

The expected moments algorithm (EMA) is an adaptation of *Schmee and Hahn*'s [1979] iterated least squares (ILS) method for fitting regression models to censored data [also see *Chatterjee and McLeish*, 1986; *Yates*, 1933; *Allan and Wishart*, 1930]. EMA and ILS are related to *Dempster et al.*'s [1977] expectation-maximization (EM) algorithm. The algorithm employs the following steps:

1. Initialization: Estimate initial sample moments, ($\tilde{\mu}_1$, $\tilde{\sigma}_1$, $\tilde{\gamma}_1$), from $\{X_S\}$.

2. EMA step 1: For $i = 1, 2, \cdots$, estimate parameters ($\tilde{\alpha}_{i+1}$, $\tilde{\beta}_{i+1}$, $\tilde{\tau}_{i+1}$) from previously estimated sample moments:

$$\tilde{\alpha}_{i+1} = \frac{4}{\tilde{\gamma}_i^2} \tag{12}$$

$$\tilde{\beta}_{i+1} = \operatorname{sign}(\tilde{\gamma}_i)\left(\frac{\tilde{\sigma}_i^2}{\tilde{\alpha}_{i+1}}\right)^{1/2} \tag{13}$$

$$\tilde{\tau}_{i+1} = \tilde{\mu}_i - \tilde{\alpha}_{i+1}\tilde{\beta}_{i+1} \tag{14}$$

3. EMA step 2: Estimate new sample moments ($\tilde{\mu}_{i+1}$, $\tilde{\sigma}_{i+1}$, $\tilde{\gamma}_{i+1}$):

$$\tilde{\mu}_{i+1} = \frac{\sum X_S^< + \sum X^> + N_H^< E[X_H^<]}{N} \tag{15}$$

where $E[X_H^<]$ is the conditional expectation of X given that $X < \Upsilon$, where $\Upsilon = \log(T)$ on the basis of current parameter values ($\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\tau}_i$). The expectation can be expressed in terms of the incomplete Gamma function, $\Gamma(y, \alpha)$:

$$E[X_H^<|\alpha, \beta, \tau] = E[X|X < \Upsilon, |\alpha, \beta, \tau]$$

$$= \tau + \beta \frac{\Gamma\left(\frac{\Upsilon - \tau}{\beta}, \alpha + 1\right)}{\Gamma\left(\frac{\Upsilon - \tau}{\beta}, \alpha\right)} \tag{16}$$

where

$$\Gamma(y, \alpha) = \int_0^y t^{\alpha-1} \exp(-t)\, dt \tag{17}$$

The P-III distribution generally has either a lower or upper bound. As a result, an observation may lie outside the support of a fitted P-III distribution, which occasionally happens when fitting systematic data by method-of-moments or L moments. Although this circumstance would suggest a lack of fit, it does not interfere with most moments-based fitting procedures. However, when this situation arises with censored data (because T, the upper bound for a censored observation, is below the lower bound for the fitted distribution), special steps must be taken, and it is not obvious how EMA should compute expected moments. In this research, moments (only for the current iteration of EMA) were computed by treating the censored observation as a systematic observation of magnitude T.

The second and third central moments are estimated by

$$\tilde{\sigma}_{i+1}^2 = \{c_2(\sum(X_S^< - \tilde{\mu}_{i+1})^2 + \sum(X^> - \tilde{\mu}_{i+1})^2) + N_H^< E[(X_H^< - \tilde{\mu}_{i+1})^2]\}/N \tag{18}$$

where

$$c_2 = \frac{N_S^< + N^>}{N_S^< + N^> - 1} \tag{19}$$

is a bias-correction factor ensuring that EMA coincides with B17 when $N_H = 0$,

$$\tilde{\gamma}_{i+1} = \{c_3(\sum(X_S^< - \tilde{\mu}_{i+1})^3 + \sum(X^> - \tilde{\mu}_{i+1})^3) + N_H^< E[(X_H^< - \tilde{\mu}_{i+1})^3]\}/N\tilde{\sigma}_{i+1}^3 \tag{20}$$

with corresponding bias correction factor,

$$c_3 = \frac{(N_S^< + N^>)^2}{(N_S^< + N^> - 1)(N_S^< + N^> - 2)} \tag{21}$$

and

$$E[(X_H^< - \tilde{\mu})^p|\alpha, \beta, \tau] = E[(X - \tilde{\mu})^p|X < \Upsilon, \alpha, \beta, \tau]$$

$$= \sum_{j=0}^{p} \binom{p}{j} \beta^j (\tau - \tilde{\mu})^{p-j} \left(\frac{\Gamma\left(\frac{\Upsilon - \tau}{\beta}, \alpha + j\right)}{\Gamma\left(\frac{\Upsilon - \tau}{\beta}, \alpha\right)}\right) \tag{22}$$

4. Convergence Test: Iterate EMA steps 1 and 2 until parameter estimates converge.

4. Performance of the Expected Moments Algorithm

Monte Carlo experiments were conducted to determine EMA's performance with various combinations of N_S, N_H, T,

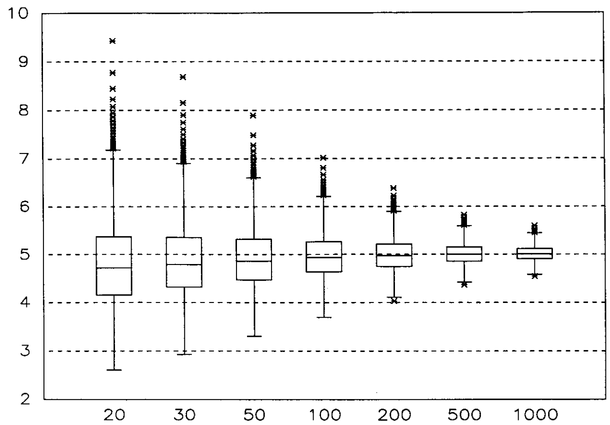

Figure 2. Distribution of $\hat{X}_{0.99}$ for $N_S = 20$ to $N_S = 1000$, with $N_H = 0$ and skew = 1.0.

and population shape parameter, α (or skew = sign (β) $\times$ $2/\alpha^{1/2}$). For each set of parameters, 100,000 replicate samples were generated and fitted. EMA converged rapidly for every sample, usually in fewer than 10 iterations.

As with *Stedinger and Cohn* [1986], performance was measured in terms of bias and variance of $\hat{X}_{0.99}$, the logarithm of the 100-year flood. $\hat{X}_{0.99}$ has two distinct advantages over $\hat{Q}_{0.99}$: $\hat{X}_{0.99}$ is invariant with respect to β and τ, so these population parameters could be arbitrarily set to $\beta = 0.5$ and $\tau = 0.0$; and the variance of $\hat{X}_{0.99}$ is nearly inversely proportional to record length, which permits summarizing results in terms of effective record length and average gain. In any case, the characteristics of $\hat{Q}_{0.99}$ can be easily inferred by imagining a logarithmic vertical axis on the box plots.

Figure 2 contains boxplots of EMA's $\hat{X}_{0.99}$ as a function of N_S when no historical information is available. As with the other figures in this paper, Figure 2 illustrates the case $\{\alpha = 4, \beta = 0.5, \tau = 0\}$ for which the true value of the logarithm of the 100-year flood is $X_{0.99} = 5.02$. Because $N_H = 0$, EMA (and B17) collapses to the standard method-of-moments estimator. EMA was found to be negatively biased for all N_S, although the bias appears to be substantial only for $N_S \leq 50$. This bias may be explained by the well-understood bias in estimating skewness from small samples [*Bobee and Robitaille*, 1977; *Lall and Beard*, 1982]. Although correcting for bias in skewness is not considered in this paper, in practice it might be desirable to do so (see *Stedinger* [1980] for discussion on whether to correct for bias).

Figure 3 plots the inverse of the variance of $\hat{X}_{0.99}$ as a function of N_S, which is seen to be nearly linearly proportional to N_S. When discussing estimators which use historical information, it is convenient to express their efficiency in terms of an equivalent number of years of systematic record. This is called "effective record length" and is denoted N_{eff}. N_{eff} can be thought of as the "value," measured in years of systematic record, of the specified combination of systematic and historical information. N_{eff} is estimated by

$$N_{\text{eff}} = 50 \frac{\text{Var } [\hat{X}_{0.99}|N_S = 50, N_H = 0]}{\text{Var } [\hat{X}_{0.99}|N_S, N_H]} \tag{23}$$

Figure 4 shows the N_{eff} of $\hat{X}_{0.99}$ as a function of total record length when $N_S = 50$. The figure contains results for five censoring threshold probabilities ($P_T = 0.000, 0.500, 0.900, 0.990, 0.999$). $P_T = 0.000$ corresponds to no censoring. When $N_H = 1000$ and $P_T = 0.990$, N_{eff} is greater than 500 years, yet only 10 historical floods are expected to be greater than T. This is a substantial increase!

For a given censoring threshold and skew, the relation between N_{eff} and N_H can be approximated by

$$N_{\text{eff}} = N_S + \lambda N_H \tag{24}$$

Stedinger and Cohn [1986] call λ the "average gain" of the estimator.

Table 1 reports λ as a function of P_T, the threshold nonexceedance probability, for six flood populations {skew = ± 0.2, ± 0.5, ± 1.0}; λ equals unity if all observations are expected to exceed the censoring threshold and zero if no observations are expected to exceed the threshold. For those cases in which some observations are expected to exceed the threshold, λ is a measure of the information content of 1 year of historical period length relative to 1 year of systematic gage record.

5. Binomial and Interval Censoring

In some cases the logarithm of a flood's magnitude is known only to lie within an interval $[Y_l, Y_u]$, where $-\infty \leq Y_l < X < Y_u \leq \infty$. This is known as "interval censoring." A special case of interval censoring, known as "binomial censoring" [*Stedinger and Cohn*, 1986], is said to occur when the exact magnitude of an historical flood is unknown except that it exceeded a lower threshold, T_l.

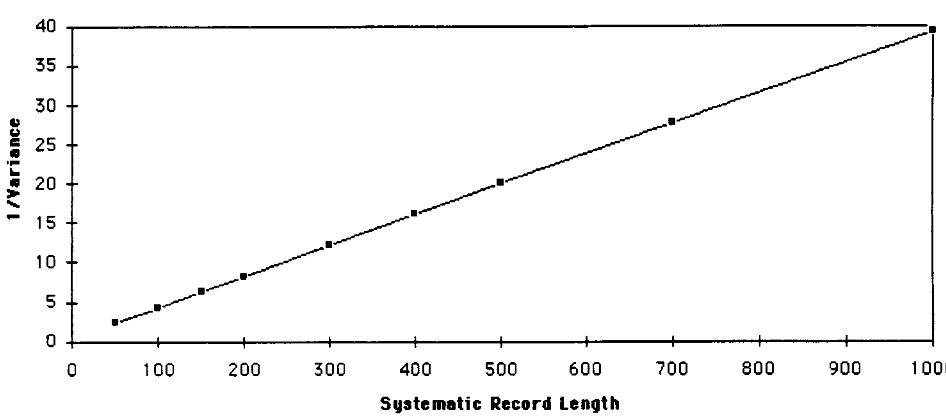

Figure 3. The inverse variance of $\hat{X}_{0.99}$ as a function of total record length, with $N_H = 0$ and skew = 1.0.

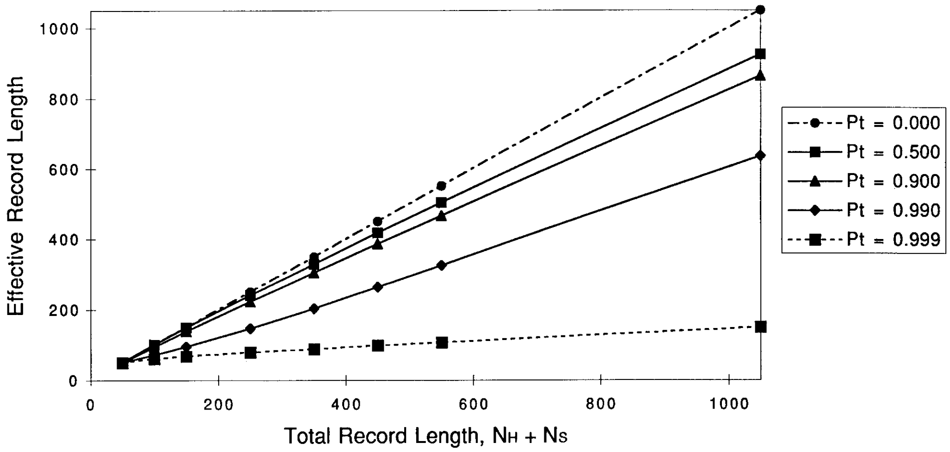

Figure 4. N_{eff} as a function of total record length and P_T, the threshold nonexceedance probability, for $N_S = 50$.

EMA can be generalized to employ binomial or interval censored data by replacing (22) with

$$E[(X-\tilde{\mu})^p|\alpha,\beta,\tau] = E[(X-\tilde{\mu})^p|\Upsilon_l < X < \Upsilon_u, \alpha, \beta, \tau]$$

$$= \sum_{j=0}^{p} \binom{p}{j} \beta^j(\tau-\tilde{\mu})^{p-j} \cdot \left(\frac{\Gamma\left(\frac{\Upsilon_u-\tau}{\beta}, \alpha+j\right) - \Gamma\left(\frac{\Upsilon_l-\tau}{\beta}, \alpha+j\right)}{\Gamma\left(\frac{\Upsilon_u-\tau}{\beta}, \alpha\right) - \Gamma\left(\frac{\Upsilon_l-\tau}{\beta}, \alpha\right)} \right) \quad (25)$$

Figure 5 shows N_{eff} for binomial-censored data when $N_S = 50$. Comparing Figures 4 and 5 for the case where the censoring threshold is at the 100-year flood line (that is, $\Upsilon = X_{0.99}$) reveals that in some cases the binomial-censored data can provides nearly as much information as the censored data. This is consistent with the findings of *Stedinger and Cohn* [1986]. However, the value of binomial-censored data declines as the distance between Υ and X_p increases. Table 2 reports the average gains for binomial-censored data corresponding to a 50-year systematic record and 200 years of historical record: the same cases reported in Table 1 for censored data. The gains are much smaller except when the threshold is at the 100-year flood line, and the average gain was found to be negative, indicating a reduction in precision, for $P_T = 0.900$ and skew of -1.0.

Table 1. Average Gains of Estimators

α	Skew	P_T 0.900	0.990	0.999
4	−1.00	0.91	0.77	0.33
16	−0.50	0.88	0.59	0.20
100	−0.20	0.81	0.35	0.14
100	0.20	0.73	0.25	0.11
16	0.50	0.73	0.38	0.13
4	1.00	0.71	0.40	0.12

Average gains are estimated by $\lambda = (50/200)\{[(\text{Var}\,[\hat{X}_{0.99}|N_S = 50, N_H = 0])/(\text{Var}\,[X_{0.99}|N_S = 50, N_H = 200)] - 1\}$. For example, the average gain for skew of -1.00 and $P_T = 0.990$ is 0.77, which means that the increase in effective record length attributable to 200 years of historical period would be about 154 years.

6. Comparisons With Other Estimators

Monte Carlo experiments were conducted to determine the relative performance of several flood-frequency estimators: (1) the expected moments algorithm (EMA), (2) the *Bulletin 17B* (B17) historical adjustment, (3) L Moments (LMOM), and (4) (local) maximum likelihood estimation (MLE). EMA and B17 are discussed above. LMOM is the standard L-moments estimator [*Hosking*, 1991] and was applied only to the case $N_H = 0$.

6.1. Maximum Likelihood Estimation

MLE estimates were derived from the standard likelihood function for censored data [*Stedinger and Cohn*, 1986]:

$$L = \prod f(X_S|\alpha,\beta,\tau) \prod f(X_H^{>}|\alpha,\beta,\tau) \binom{N_H}{N_H^{<}} F(\Upsilon|\alpha,\beta,\tau) \quad (26)$$

where $F(\ \ |\alpha, \beta, \tau)$ is the cumulative distribution function corresponding to $f(\ \ |\alpha, \beta, \tau)$. A quasi-Newton method (*International Mathematical and Statistics Library, Inc.* (*IMSL*) [1987] subroutine DBCONF) was used to search for the maximum of L. The method failed to find a local maximum for approximately 5% of the generated samples when $\alpha = 4$. In these cases the MLE outcome was discarded, which may have produced bias in the MLE box plots because they are based on a (nonrandom, 95%) subset of the Monte Carlo samples. For $\alpha > 4$ the MLE failure rate was much higher. To avoid comparisons where the MLE failed to converge, the comparisons among estimators considers only the case $\alpha = 4$ and relatively large sample sizes.

6.2. Frechet-Cramer-Rao Bounds

Frechet-Cramer-Rao (FCR) bounds [*Condie and Pilon*, 1983; *Pilon and Adamowski*, 1993] were computed because they provide an estimate of the asymptotic variance of MLE estimators and can provide a lower bound on the variance of an estimator if certain regularity conditions are satisfied. *Harter and Moore* [1967] discuss regularity conditions for the P-III

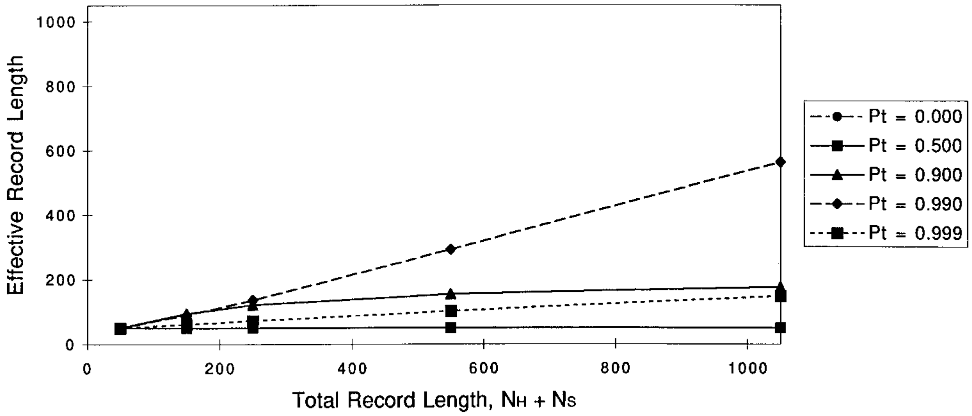

Figure 5. N_{eff} of binomial-censored data estimator as a function of total record length and P_T, the threshold nonexceedance probability, for $N_S = 50$. Note: The line corresponding to $P_T = 0.000$ is obscured by the $P_T = 0.500$ line.

distribution. The FCR bound on the variance of $\hat{X}_{0.99}$ is given by

$$V_{\text{FCR}} = \underline{G}'\underline{I}^{-1}\underline{G} \tag{27}$$

where

$$\underline{I} = -E\begin{bmatrix} \frac{\partial^2 \ln(L)}{\partial \alpha^2} & \frac{\partial^2 \ln(L)}{\partial \alpha\, \partial \beta} & \frac{\partial^2 \ln(L)}{\partial \alpha\, \partial \tau} \\ \frac{\partial^2 \ln(L)}{\partial \alpha\, \partial \beta} & \frac{\partial^2 \ln(L)}{\partial \beta^2} & \frac{\partial^2 \ln(L)}{\partial \beta\, \partial \tau} \\ \frac{\partial^2 \ln(L)}{\partial \alpha\, \partial \tau} & \frac{\partial^2 \ln(L)}{\partial \beta\, \partial \tau} & \frac{\partial^2 \ln(L)}{\partial \tau^2} \end{bmatrix} \tag{28}$$

and

$$\underline{G} = \begin{bmatrix} \frac{\partial X_{0.99}}{\partial \alpha} \\ \frac{\partial X_{0.99}}{\partial \beta} \\ \frac{\partial X_{0.99}}{\partial \tau} \end{bmatrix} \tag{29}$$

The partial-derivative terms are given by *Harter and Moore* [1967]. To represent the FCR results as a box plot along with the samples obtained by Monte Carlo analysis of the other estimators, a pseudo-FCR sample was generated to represent a normally distributed and unbiased estimator with the FCR variance. The pseudosample values were

$$X_{\text{FCR},i} = X_{0.99} + V_{\text{FCR}}^{1/2}\Phi^{-1}\left(\frac{i}{10001}\right) \tag{30}$$

$$i = 1, \cdots 10000$$

where $\Phi^{-1}(\ )$ is the inverse of the standard normal cumulative distribution function.

6.3. Monte Carlo Results

Ten thousand random samples were generated. As in the previous experiments, the performance of each estimator was measured in terms of the bias and variance of $\hat{X}_{0.99}$.

Figure 6 uses box plots to compare MLE, EMA, B17, and LMOM for $N_S = 30$ and $N_H = 0$. The variance of each estimator slightly exceeds the FCR bound. Only EMA and B17, which are identical when $N_H = 0$, exhibit substantial bias.

Figures 7 and 8 compare FCR, MLE, EMA, and B17 for $P_T = 0.90$ and $P_T = 0.99$, respectively, when $N_S = 30$ and $N_H = 1000$. The expected number of threshold exceedances, $N_H^{>}$, is 100 for the case illustrated in Figure 7. Although these

Table 2. Average Gains of Binomial Censored Data Estimators

		P_T		
α	Skew	0.900	0.990	0.999
4	−1.00	−0.03	0.22	0.06
16	−0.50	0.10	0.39	0.11
100	−0.20	0.18	0.38	0.13
100	0.20	0.24	0.36	0.11
16	0.50	0.25	0.33	0.10
4	1.00	0.24	0.33	0.09

Average gains are estimated by $\lambda = (50/200)\{[(\text{Var}\,[\hat{X}_{0.99}|N_S = 50, N_H = 0])/(\text{Var}\,[X_{0.99}|N_S = 50, N_H = 200)] - 1\}$. For example, the average gain for skew of -1.00 and $P_T = 0.990$ is 0.22, which means that the increase in effective record length attributable to 200 years of historical period would be about 44 years.

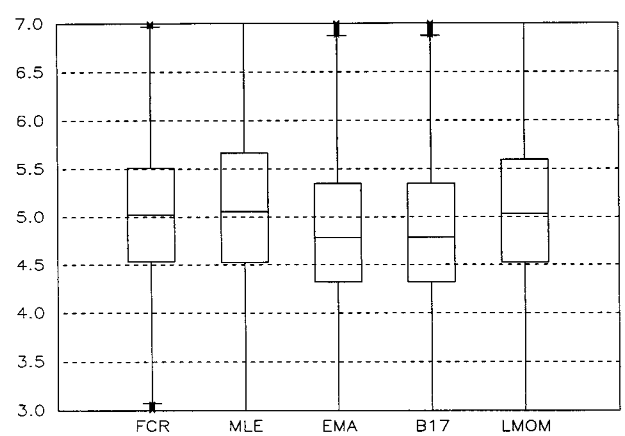

Figure 6. Distribution of $\hat{X}_{0.99}$ for FCR, MLE, EMA, B17, and LMOM when $N_S = 30$ and $N_H = 0$.

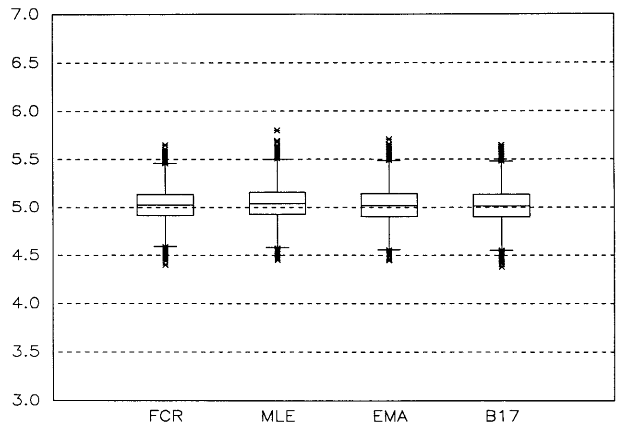

Figure 7. Distribution of $\hat{X}_{0.99}$ for FCR, MLE, EMA, and B17 when the threshold nonexceedance probability $P_T = 0.90$, $N_S = 30$, and $N_H = 1000$.

figures illustrate the case of an extremely rich data set, historical flood data extending back several hundred years are not unusual [*Stedinger and Baker*, 1987; *O'Connor et al.*, 1994]. In this case the estimators perform nearly identically. However, where $E[N_H^>]$ is smaller, as in Figure 8 where $E[N_H^>] = 10$, B17 exhibits substantially greater variability, approximately four times higher variance, than do the other estimators. This result is of practical significance because the threshold corresponding to paleoflood data is often at or above $X_{0.99}$.

7. Conclusions, Cautions, and Future Research

EMA offers a straightforward method for incorporating historical flood information into flood frequency studies. By making use of threshold-exceedance information, EMA achieves greater efficiency than the B17 adjustment, nearly achieving the efficiency of maximum likelihood estimation while avoiding MLE's numerical complications. Because EMA employs the method of moments, it is compatible with all of the features of the current B17 guidelines. Future work will need to address development of confidence intervals for design events, among other issues.

One must be cautious, however. EMA employs types of information (knowledge of the discharge corresponding to an exceedance threshold, the exact number of threshold exceedances, and the value of N_H) that have not traditionally been collected. Our ability to provide such data for historical periods, and to assure their quality, has yet to be established. Use of unreliable historical information may degrade rather than improve flood-frequency estimates [*Hosking and Wallis*, 1986b; *National Research Council*, 1988; *Kuczera*, 1992].

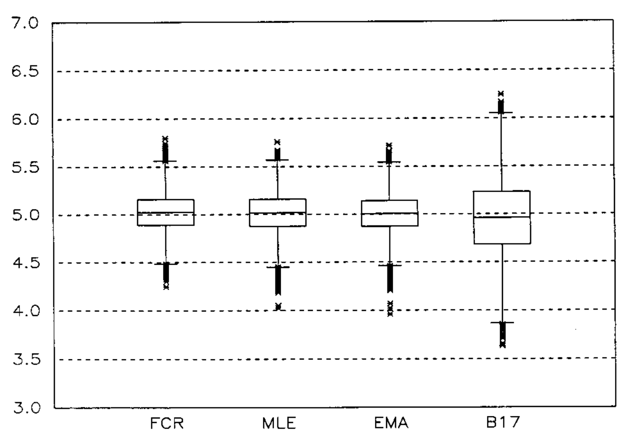

Figure 8. Distribution of $\hat{X}_{0.99}$ for FCR, MLE, EMA, and B17 when the threshold nonexceedance probability $P_T = 0.99$, $N_S = 30$, and $N_H = 1000$.

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